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High-Attaining Year 10 pupils' conceptions and learning of proof: A critical analysis.

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Abstract

For most mathematicians, proof is seen as an integral part of the subject; it ensures absolute certainty and separates this subject from other sciences. Yet my own experience suggests that proof is not incorporated into the mathematics curriculum and therefore pupils are being deprived of this essential, and fascinating, cornerstone of mathematics. This study will outline the findings of a small-scale case study conducted mainly with high attaining year 10 pupils from a rural 11 – 16 comprehensive community college. Year 7 and year 10 pupils were initially given a diagnostic questionnaire. A short lesson sequence was then taught to the year 10 group and a concluding questionnaire administered. The main conclusion of the study is that the pupils taking part in this research do indeed have very little interaction with proof throughout their compulsory mathematics career. Even after changes to the mathematics curriculum that explicitly incorporate proof at all ages and levels, pupils have very little conception of proof and struggle with proof related tasks. To combat these difficulties, I argue that proof needs to be the heart of mathematics education and outline further research that can investigate the best methods to achieve this.

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"Analogy cannot serve as proof" (Louis Pasteur)

Introduction

This assignment will critically evaluate the role of proof in the mathematics curriculum and discuss what place it has in the curriculum, and what place it should possibly have in the future. These factors will be analysed by looking at the current conceptions pupils hold about proof and what their level of ability is in identifying and creating proof.

First there will be an examination of literature regarding the teaching of proof in schools. This discussion will seek to provide arguments for and will begin to discuss why proof is an integral part of the curriculum. There will be a brief discussion of what constitutes a proof and how we can best encourage a movement from informal analogous proofs to a formulation of formal proofs. The section will then look at specific studies into pupil conceptions and will conclude with the research questions that have emerged from a combination of these studies and own professional interest.

The research method for this assignment will then be detailed: a case study predominantly carried out with a high attaining year 10 class, with some comparison data collected from a high attaining year 7 class. The specific research and data collection methods will then be described and justified.

The remainder of the assignment will then be concerned with analysing the data and discussing the implications of this. Three separate questions will be evaluated and there will be a discussion of the research as a whole. The main conclusion of this project is that pupils have very little interaction with proof throughout their mathematical career and as such are missing an integral part of the subject. Whilst the research findings were not conclusive, it seems that providing lessons with an explicit focus on proof is one way to increase understanding. Future possible areas of research are then suggested.

The main conclusion of this assignment is that, even after curriculum changes that incorporate proof into mathematics education, pupils still have little conception of proof and are disadvantaged when it comes to providing proof tasks. My belief is that proof is not incorporated to its full potential. To combat this I contend that proof needs to be at the heart of the curriculum. How this should be done cannot be described by one piece of research, but further studies will be able to discuss this in more detail, and will be able to analyse any links between incorporating proof into lessons and general mathematical attainment of a class.

Literature Review

This section introduces the main literature on the role of proof in mathematics education, discussing particular issues to be investigated through research.

In the seminal text "The Nature of Proof" (1938), Harold Fawcett discusses the shift to teach geometry through deductive proof, rather than a list of memorised theorems. The implication is that understanding is desirable but not necessary; whether this is the view of the author is unclear. Rhetoric used seems to imply that providing knowledge of mathematics, and an appreciation of proof, cannot occur at the same time. The idea of two separate parts of mathematics has been a problem; pupils can either be taught an appreciation of proof or the relevant knowledge but not both. Pupils now experience a curriculum where there is a focus on knowledge and facts but little appreciation for mathematical proof and deductive argument; Ofsted (2006) mention this as a factor that acts against high achievement, motivation and participation in 14 - 19 mathematics by commenting that mathematics has become "an apparently endless series of algorithms [...] rather than a coherent and interconnected body of knowledge" (Ofsted, 2006, p.15).

Defining formal proof

First there is a need to consider what constitutes a proof and why it should be present in the mathematics curriculum. Balacheff (1988) categorises a four stage hierarchy:

1. Naïve empiricism: truth is asserted after verifying several cases.
2. The crucial experiment: a proposition is verified on one particular, non-trivial case that is typical of others.
3. The generic example: reasons for the truth of an assertion made explicit through use of a typical case.
4. The thought experiment: operations and foundational relations of the proof are indicated.

As this is a hierarchy, a proof should move down this list, towards the 'thought experiment' stage, for it to become a formal mathematical proof. To progress to the final stage, Balacheff (1988) believes three things need to occur:

1. Decontextualisation: move from one object to a class of objects.
2. Depersonalisation: move to an independent viewpoint.
3. Detemporalisation: move away from the operations occurring in their particular time.

I agree these need to occur for a proof to be considered formal, applying these concepts to a proof will make it generally true in all situations. However, I believe this formalisation could very well be taught after pupils acquire an initial appreciation for problem solving. This is an important issue in mathematics education: advanced mathematics is concerned with formal proofs, not informal sketches of proofs. I believe pupils need to first gain an appreciation of proof through informal methods, developing relevant skills, then later creating formal proofs. This argument will be developed later. Firstly, I shall discuss why formal proof should be in the mathematics curriculum.

The purpose of proof

Watson (1980) states "conjecture and proof have a central place in mathematics teaching" (Watson, 1980, p.163) but the act of proving can only be conducted in combination with understanding the methods and content of mathematics. Recently there has been too much focus on this aspect of mathematics. There has been a passive attitude to mathematics learning and teaching wherein pupils are taught to think about finished structures and often complete trivial activities around these. Ofsted (2008) make detailed comments on effective planning following observations of 192 maintained English schools; in particular they discuss that good curriculum planning provides "pupils with opportunities to apply mathematics to a variety of tasks, enabling pupils to choose approaches, reason and refine their thinking in the light of their solutions" (Ofsted, 2008, p.49). However, they found that schemes of work often had a focus "on content rather than pedagogy" (Ofsted, 2008, p.25) and that this had the effect of providing "limited opportunity for independent thought or for making generalisations, a crucial element of behaving mathematically" (Ofsted, 2008, p.50)

Watson calls for more emphasis on the "'messy' stage of thinking" (Watson, 1980, p.164), where pupils perceive that mathematics requires work, instead of appearing as a fully formed theorem and proof. Without this, pupils consider mathematics as a stream of exercises, with a solution clearly signposted from the outset. One suggested method is to encourage pupils to form conjectures: stating their own ideas, then attempting to prove or refute them, establishing a feel for proof. This increases pupils' motivation as they begin to prove themselves right, or wrong; the act of formal proof becomes a natural next step. Conducting these methods from an early age directly addresses this, placing proof at the heart of mathematics education. Stylianides and Stylianides (2009) discuss possible solutions to improve students' ability in proof, suggesting activities that not only allow an expression of mathematics but also an evaluation of these ideas, allowing pupils to see the broader implications. Thus, there needs to be a shift away from the authoritarian method of teaching and a move towards pupils becoming actively involved in the process of validating statements.

Stylianides and Stylianides (2009) believe that many teachers are uncomfortable teaching proof due to inadequate understanding. This can lead to reinforcing misconceptions such as the idea that demonstration, or empirical evidence, is an accurate substitution for proof. Two considerations need to be made when thinking about proof in a classroom setting: "mathematics as a discipline and students as mathematical learners" (Stylianides and Ball, 2008, p.309). These need to be balanced in education, yet they conflict and compete for classroom time. I believe the current curriculum favours the latter greatly; the focus on subject matter means that proof is often neglected. Stylianides and Stylianides (2009) discuss the implications of leaving proof until late in mathematics education, so when it is first encountered, it seems "alien rather than a natural extension of things they have already learned" (Stylianides and Stylianides, 2009, p.238).

Why is proof important? Watson (1980) believes one single reason is enough for us to trust that a pattern works: whilst intuition is a powerful tool in proving, "only a proof gives a completely convincing reason" (Watson, 1980, p.165). A proof allows us to see why something is true, the conditions needed for a statement to be true, and provide a deeper understanding than intuition alone. There needs to be a careful balance: focus on rigour should not be at the expense of the initial intuition and mathematical insight. MacDonald (1973) discusses the value of "heuristic proof" (MacDonald, 1973, p.103) where pupils can intuit a selection of statements is consistent and a theorem is valid, before proving it formally. In early stages of mathematics I believe that these are the important skills, the formal method can follow later. Pupils should leave school with a range of mathematical skills but I believe these should not be the focus of the curriculum and teaching, rather the by-product of an education system valuing original thought, logical argument and rigour. Jeffrey (1977) takes a more radical view and states most teachers will lead pupils through a "relatively restricted number of skills" (Jeffrey, 1977, p.15). He suggests that teachers should encourage pupils to "think of mathematics in terms of an act of mathematising rather than an accumulated body of mathematical knowledge" (Jeffrey, 1977, p.15).

Bell (1976) discusses three related functions for proof: verification, illumination, and systematisation. Coe and Ruthven (1994) explore this first function, stating the most important function of proof is to "provide grounds for belief" (Coe and Ruthven, 1994, p.42). The second function is to ensure a learner understands; a deeper insight into why a result is true allows one to make sense and meaning. The act of proving in the classroom is only useful if this insight is provided: proof is only conducted to ascertain if something is true or false. As Manin (1977) states, "A good proof is one which makes us wiser" (Manin, 1977, p.49). The third function of proof is to "exhibit the logical structure of ideas" (Coe and Ruthven, 1994, p.42). Personal understanding comes through observation and intuition, forcing a particular structure onto all proofs means the argument is clear and precise for all. Mathematics is to be communicated, something often not made explicit in education. Ofsted (2006, 2008) have made many comments on this in recent years and it has been discussed that "effective teachers required students to articulate and refine their ideas" (Ofsted, 2006, p.9) and that pupils being able to express their mathematics in formal languages meant that they were "able to learn very effectively from one another" (Ofsted, 2006, p.12). However, it was also found that "most lesson do not emphasise mathematical talk enough; [and] as a result, pupils struggle to express and develop their thinking" (Ofsted, 2008, p.5).

In 1999, the National Curriculum (Department for Education and Employment and Qualifications and Assessment Authority, 1999) underwent significant changes. In mathematics, proof was acknowledged explicitly at all Key Stages, as opposed to only being available as extension material for the most able. At Key Stage 1, pupils are expected to utilise "explanation skills as a foundation for geometrical reasoning and proof in later key stages" (DfEE, 1999, p.39). Key Stage 2 builds on this to "develop logical thinking and mathematical reasoning" (DfEE, 1999, p.42). At Key Stages 3 and 4, pupils are taught to "develop short chains of deductive reasoning and concepts of proof" (DfEE, 1999, p.50) and "distinguish between practical demonstration, proof, conventions, facts, definitions and derived properties" (DfEE, 1999, p.53). Extension work for high-attaining pupils is to develop key properties of proof and an "appreciation and explanation of how more complicated properties and results can be derived from simpler properties and results" (DfEE, 1999, p.72).

It is my belief that proof is not effectively utilised; it should be at the very heart of mathematics education. I want to investigate further pupil conception of proof. Studies such as Healy and Hoyles (1998) and Coe and Ruthven (1994) were conducted before changes in the Curriculum, at a time when proof was rarely taught. Pupils now have been taught proof throughout their educational career and I will investigate if there are any differences in views and ability to identify and construct proofs. My belief is years 7 and 10 have similar abilities in proof identification and creating, despite the three year gap. My albeit limited experience as a trainee teacher has shown proof is barely acknowledged in secondary school education so I expect both ability and perception of proof to be similar. I would go as far to say year 7 pupils may have more appreciation of the use of proof and its purpose as there has recently been a shift in primary school mathematics to discovery and investigational mathematics, where links to proof are more prevalent.

Research on proof in education

Coe and Ruthven (1994) conducted an analysis of previous studies into students' understanding and use of mathematical proof. The results and comments in these investigations could be placed under three headings:

- Intellectual functions of proof
- Student perceptions of proof
- Student progression in proof

The first point has already been discussed, the second and third will be the focus of this analysis.

Perception of proof is discussed in detail by Schoenfeld (1985) who states the belief system of a student is crucial in understanding the way they do, or do not, use a proof. Pupils may have access to knowledge required to construct a proof, yet Schoenfeld (1985) found these elements were often not used as pupils "did not perceive their mathematical knowledge as being useful to them, and consequently did not call upon it" (Schoenfeld, 1985, p.13). This was echoed by Healy and Hoyles (1998) who found pupils recognised the importance and generality of a proof, but would then fail in distinguishing and describing relevant properties, therefore would rely on empirical verification. Schoenfeld (1985) further found most students viewed proof as a purposeless activity where the objective is to confirm the intuitively true. Consequently, pupils were found to view proof as a redundant exercise, with no sense of the enlightenment and understanding it can provide. Balacheff (1991) discusses possible reasons for these views: in the classroom, pupils act not "as theoreticians but as practical persons" (Balacheff, 1991, p.187). Cobb (1986) distinguishes between 'self-generated mathematics' and 'academic mathematics': the former involving discovery, the latter learning facts about past discoveries. Both undoubtedly important, but there is an imbalance in teaching and too much focus on the latter; Ofsted (2006) found that there were many occasions where teachers would present "mathematics as a collection of arbitrary rules and procedures, allied to a narrow range of learning activities in lesson which do not engage students in real mathematical thinking" (Ofsted, 2006, p.5). This was later described as an approach to teaching that "fragments the mathematics curriculum" (Ofsted, 2008, p.37) and therefore the view that mathematics is about efficiency and fact recall, rather than rigour and reasoning, is reinforced.

The point of student progression in proof from Coe and Ruthven (1994) is explored by Porteous (1990) who states that "proof types used by children are naturally informal, but it would be a mistake to devalue them because of this" (Porteous, 1990, p.597). Explaining generalities should be encouraged and discovery skills will then become second nature for pupils. It is my belief that pupils should explore mathematics with less of an emphasis on rigour and abstraction; aspects to be introduced in later stages, once the skills have been mastered. Pupils should discover first and formalise later; they should be encouraged to be creative, without fear of being wrong or displaying work in a formal manner.

These ideas are the motivation for the research method, discussed in the following section. I believe that, even a short lesson sequence on proof, and the thinking behind it, will create changes in pupils' ability to form proofs and their conceptions of this area. Lessons should combine activities, allowing for creative thinking and encouraging logical reasoning with arguments and proofs.

In order to find out the level of proof pupils were performing at, Coe and Ruthven (1994) analysed work from students using the hierarchies. They identified where proofs were used and classified them as empirical, weak deductive or strong deductive. They found that, after following the School Mathematical Project module on problem solving, pupils still struggled to grasp the idea of proof and how it is formed. Only 2 out of 60 pieces of work contained a strong deductive proof, with only a further 4 containing a weak deductive proof. Related to this is the idea of a "proof opportunity" (Coe and Ruthven, 1994, p.45), where an explanation was sufficient to be understood and demonstrated knowledge of the situation, yet was not extended to a formal proof. This suggests requisite mathematical knowledge is present, yet pupils do not have a strong enough concept to form their own proof.

Coe and Ruthven (1994) acknowledge an issue I consider important: they found few pieces of work demonstrated a "need to explain why" (Coe and Ruthven, 1994, p.47), suggesting that pupils have little appreciation of mathematics being about discovery and how crucial it is to question statements and seek explanations. Echoing this, Healy and Hoyles (1998) discovered most able year 10 students were "unable to distinguish and describe mathematical properties relevant to a proof and use deductive reasoning in their arguments" (Healy and Hoyles, 1998, p.6). Furthermore, they found students had a "lack of familiarity with the process of proving [and] little idea of this process" (Healy and Hoyles, 1998, p.6). This implies that students need to be encouraged to appreciate the power of justifying all statements and understand mathematics is not just 'getting the right answer'. As previously discussed, there is too much focus on facts in mathematics education and when problem-solving activities are conducted, we expect pupils to be able to ask why and investigate properly but they have not been taught the relevant skills needed; "too often, pupils are expected to remember methods, rules and facts without grasping the underpinning concepts [...] and making sense of mathematics so that they can use it independently" (Ofsted, 2008, p.5). As such, pupils need to be allowed to access discovery activities where they can generalise statements and ideas, then make the final move towards formal proof.

From my examination of the literature, I believe four important questions require further investigation:

1. What conception do high attaining Year 7 and Year 10 students have about proof?
2. How successful are Year 10 students at identifying and creating proof arguments?
3. Does a short lesson sequence on proof have an effect on the previous conceptions of proof, and the ability to identify and create proof arguments in pupils?
4. Are there any links between Year 10 students' conceptions on proof, and their ability to identify and form proof arguments?

What follows is an account of the research aimed to address these questions and to provide some evidence that the teaching of proof does improve ability to create arguments and create more informed pupils.

Research Methods

This section discusses research methods to be used in this assignment, evaluates why they were chosen and how they are appropriate to answer the research questions. There will be an outline of why this data is important and how it will be analysed. There will also be an account of ethical procedures followed for this study.

The research method of this assignment is in four separate parts, primarily conducted with a high attaining year 10 class I had already worked with in a rural 11 – 16 comprehensive community college. I wanted pupils to be able to focus on proof and not struggle over subject matter, possibly an issue if this was conducted with earlier years where knowledge would be of a lower level. The first activity was a diagnostic questionnaire based on the material used by Healy and Hoyles (1998) given to both a year 7 and year 10 class. This allows both a comparison of conceptions of proof from two different year groups and also assesses the level of year 10 pupils' knowledge of proof and their ability in identifying and constructing proofs. Results were compared to see if there were any clear differences or trends emerging from this. After analysis, 2 lessons were devised for the year 10 class, looking in more detail at proof. They also examined how to work through the creation of a proof for a particular conjecture, similar to those in the original test. Finally, year 10 was given another follow up test, very similar to the original test, asking questions about activities similar to those discussed in lessons.

The reason behind such a varied and detailed approach is that, even though a case study approach is being taken to answer the research questions, I feel collecting a variety of data in different ways helps overcome some of the criticisms of this approach. Bell (2005) states a case study occurs when "evidence has to be collected systematically, the relationship between variables studied [...] and the investigations methodically planned" (Bell, 2005, p.10). However, the nature of looking at one specific class means that it is difficult to cross check information and generalise results. Investigating two classes could overcome this and create more valid results. Another reason for conducting a variety of methods is best described by Taber (2007: 76): "the collection of different types of data provides the basis for an in-depth qualitative analysis". As such, a 'triangulation' of results should lead to more accurate conclusions. There does need to be an awareness of the "degrees of freedom" (Taber, 2007, p.76) in what is being studied; asking the same question in different forms provides no more valid data than asking it once. As such, the tests and lessons will contain a range of questions asked over different contexts, providing more pieces of data that can be combined to form a larger picture.

Research Design

As mentioned, the two diagnostic tests were based on the tests administered by Healy and Hoyles (1998) and Küchemann and Hoyles (2006, 2008). The purpose of their questionnaires was to ascertain student views on the role of proof and look at the ability of students in identifying and creating proofs. Appendix 1 contains the initial test administered in this assignment; questions A1, A2, A3, A6, G1, G2 and G3 are full or partial replications of questions from Healy and Hoyles (1998, pp.82 – 103). Question G6 is a replication of a question from the Year 8 Proof Survey created by Küchemann and Hoyles (2006, 2008).

The test started with an open question, allowing pupils to write everything they know about proof. This is then used to investigate the pupils' views of proof. Subsequent questions are concerned with identifying and constructing proof. Firstly, pupils were presented with a statement and a choice of proofs either supporting or refuting the claim. From these arguments students choose the one closest to their own approach, and the one they believed would receive best marks from their teacher. Pupils were also asked to grade the questions based on their accuracy and situations when the proof would be correct. After being presented with a variety of similar proofs and questions, pupils were asked to construct a proof of their own.

The planning of the short lesson sequence followed an analysis of the questionnaires. The two lessons involved working through an activity book, which was then taken in to analyse. The focus of the first lesson was geometric proofs, specifically involving triangles. Pupils were taught about the importance of rigour, being able to sift the information we have and manipulate it to form a proof for something we want. Pupils also looked at the idea of proving converses and the logical implications of proving the converse of a theorem. The second lesson looked at numerical proofs, where the focus was on moving from pictorial representations to algebraic solutions. Pupils then looked at a false proof and tried to identify mistakes in a proof that seemed correct at the first glance. Having an appreciation of questioning every step of a solution is important and helps to question oneself when writing proofs. Inspiration for the lesson sequence is taken from Sue Waring's book (2000) and the proof writing frame from lesson 1 is a replication of material included in the book (Waring, 2000, p.104). Lessons would extend the material from the tests, so the subject matter was inspired by the work of Healy and Hoyles (1998) and Küchemann and Hoyles (2006, 2008).

The two lessons were planned after results from the first test had been analysed. The following section discusses the results in detail but the ability level in proof was low so lessons were planned assuming no knowledge of proof whatsoever. Furthermore, the class, even though generally of high attainment, was still mixed so the pre-requisite mathematical knowledge was also low; the focus of these lessons was proof, not testing previous subject knowledge.

The second questionnaire is very similar in format to the first; the differences lie in the material (see Appendix 2). In this second survey, questions asked were possibly more demanding in places and more tailored to material covered in the lessons. There was a repeat of two questions from the first survey; written proof questions used to analyse any changes in approach to writing proofs and possibly even an improvement in ability. Question G6 is a replication of a question from the Year 8 Proof Survey; questions G1, G3 and A1 are taken from the Year 9 Proof Survey; and questions G3 is taken from the Year 10 Proof Survey. These three surveys are all taken from the Proof Materials Project created by Küchemann and Hoyles (2006, 2008). Question A6 is taken from the questionnaire created by Healy and Hoyles (1998, pp.82 – 103).

In the initial planning stages of this research, there was going to be a semi-structured interview conducted, to collect further data and strengthen claims made in later sections (Taber, 2007). Whilst the absence of an interview could weaken the data and results of this assignment, I believe further provisions in lessons combats this and there can still be an accurate comparison between the year groups and effective suggestions for methods to improve the level of ability in proof construction. Utilising group working methods meant that the benefits of group interviews are still present in the research design; Taber (2007: 156) describes these benefits as being when "comments of one student to act as a stimulus for another, perhaps eliciting information that would not otherwise have been revealed".

Ethical Considerations

One of the most important considerations when conducting research is being ethical and protecting participants from any type of harm. In relation to this research the harm is emotional and mental, rather than physical. In my opinion, it is most important to be completely open with participants, informing them of the purpose of the research and what their data will be used for. Denscombe (2007) describes methods that can be employed to ensure ethical practice and a selection of these will be utilised. Before the tests and the interview, pupils were informed of the full nature of the research and how the tasks would fit into this. Pupils were also informed that their tests, and any resulting data, would be anonymous; the only identification is a numbering system; tests will only be available to me and their usual teachers will not be able to see them. Appendix 3 contains the statement that was read out before questionnaires were completed. Pupils were able to ask further questions into the nature of the survey and the research in general. One consideration was to constantly reinforce the idea that all data is anonymous. This is important not just in terms of ethics but also led to more reliable results (Cohen, 2007).

Bell (2005) summarises the situation as this: "ethical research involves getting the informed consent of those you are going to interview, question, observe or take materials from" (Bell, 2005, p.45). Pupils could opt out of any of the tests and were also able to leave the research at any time. The lesson sequence was taught to everyone as this formed part of their usual mathematics lessons, and little data was taken from these lessons. Appendix 4 shows a short code of practice that I wrote to follow throughout this research. It takes inspiration from a number of sources, predominantly Bell (2005, p.51).

Results and Discussion

This section discusses the results of carrying out the above data collection methods and provides an insight into how data can be interpreted. This incorporates relevant literature, begins to answer the original research questions, and discusses how findings correlate with other studies.

The first survey with year 10 was conducted on 8th March 2011, and with year 7 on 11th March. The short lesson sequence was taught on 29th March; 30th March; and 5th April. The final lesson also contained the second survey with year 10. Appendices 5 and 6 contain exemplar pages of how the survey results were displayed and analysed.

In order to begin to answer the research questions, the data needs to be looked at in 3 ways:

1. Year 7 and year 10 responses to question P1 from the first questionnaire will be discussed.

2. Data from the first and second questionnaire, conducted by year 10 students, will be discussed.

3. Year 10 conceptions of proof and ability in forming and identifying needs to be compared to identify any possible relationships.

Comparison of Year 7 and Year 10 responses to P1

> Question P1:
>
> You are going to complete survey that is all about proof.
>
> Before you start, write everything you know about proof in mathematics and what you think it is for.
>
> If it helps, you can use examples, pictures or even a story: just as long as it's about proof!
>
> (Appendix 1, p.2)

Answers from year 7 pupils when asked about their conceptions of proof are remarkably similar to each other. There were 33 respondents and, of these, 14 explicitly mentioned that proof is evidence for a statement. Some of these pupils may be aware of the word from other sources and the link between proof and evidence may not be from prior mathematical knowledge. However, some expanded their explanations and went on to provide an example of a proof.

One of the main points is that year 7 pupils seem to understand the purpose of proof more than year 10. Instead of proof being a routine operation that tells us nothing, pupils seem to understand proof is an important part of mathematics. One particular pupil writes that proof is "a bit like working things out in maths and having to show them [sic] workings [...] proving what you think is right" (Year 7 Pupil 10). This implies that some year 7 pupils are aware that proof alone can provide you with absolute certainty: intuition is not enough. In comparison over half (7/13) of year 10 students admitted they had no idea what proof was. This does not necessarily demonstrate a complete lack of knowledge of proof, as it is possible that they have completed proof activities without understanding it was proof.

The remaining 6 students differed in their certainty of proof but all seemed to suggest the purpose is verification. There is a similarity here with year 7 responses. It seems year 7 students have a different conception of proof itself, possibly understanding the usefulness and purpose more than year 10 who possibly view proof as a purely administrative task. It appears year 10 students consider the purpose of proof as justifying an already constructed statement. There is no concept of its usefulness as a deductive activity and there does seem to be a suggestion that proof is an unimportant extraneous activity. One conclusion may be that year 7 students have a greater conception of proof than those in year 10. This suggests that there has been little improvement in teaching of proof since the changes to the National Curriculum in 1999. However, there is no real insight into why this is. Is it just that year 7 are more open to the idea of proof; able to make connections with other areas of thinking or other subjects?

Coe and Ruthven (1994) conducted interviews with pupils to find their views on proof. The most prevalent view was that proof is a sort of check to show a statement is absolutely correct. However, pupils did view there being a split in mathematics between a written proof and ‘normal’ mathematics. Formal proof is seen as an activity conducted in a particular context, not something to be carried out for its own benefits. One pupil commented there is a distinction between intuition, described here as a ‘mental proof’, and a ‘formal proof’ that must be written down. This implies lack of appreciation that proof can provide understanding; instead proof is viewed as an administrative task to be completed after you have found something. Healy and Hoyles (1998) made a particularly relevant statement: “students are most likely to describe proof as about establishing the truth of a mathematical statement, although a substantial minority ascribe it an explanatory function and a further large number have little or no idea of the meaning of proof and what it is for” (Healy and Hoyles, 1998, p.18). This is something partially echoed in the above results: many year 7 pupils did indeed describe proof as being used to give evidence for the truth of a mathematical statement, something less prevalent in year 10 responses. Many pupils from both years have no idea what proof, or it’s function, is. No pupils mentioned the explanatory function of proof and there was no discussion on proof providing an insight into a statement or an area of mathematics.

After the sequence of lessons, year 10 pupils were again asked about conceptions of proof. In this second survey, pupils seemed to demonstrate a greater awareness of what proof is. There were more answers discussing the purpose of proof and what it involves; 5/12 responses seemed to show some sort of appreciation of proof and it's purpose. Some pupils did still believe proof is just something to do with algebra or how you demonstrate the state of matters; this could be something that could be remedied with further lessons or discussion on proof. In general these results do seem to show that lessons in proof and incorporating proof into mathematics teaching has an effect on pupils and creates a situation where pupils consider proof as being an absolute explanation of things that seem intuitively true.

Comparison of Year 10 data from Surveys 1 and 2

Question A1 concerns proofs of the statement "When you add any 2 even numbers, your answer is always even" (Appendix 1, p.3). In year 10 responses from the first survey, pupils were split between selecting either algebraic or narrative proofs as best answer to the statement. However, pupils were then more likely to select an inaccurate empirical proof for their own answer. Interestingly, when pupils were asked which answer they thought their teacher would prefer, over half (7/13) chose an answer more resembling algebra. This is something that continued to be demonstrated in the second survey where 7/13 pupils selected an algebraic or narrative approach when asked about the best approach to proving the statement. Even after lessons on proof, 10/13 pupils selected an empirical approach to prove the statement. Pupils still were of the opinion that a teacher would want to see an algebraic approach as a proof; 8/13 pupils stated this as the teacher's preference. This echoes findings by Healy and Hoyles (1998) who stated "students believe that proving [...] by means of a formally-presented analytic argument will receive the best mark" (Healy and Hoyle, 1998, p.22). What was expected was that students would not change their opinion of a teacher's requirements of proof but would change their own view of what constitutes a valid proof. This would have meant a higher response rate for the analytic proofs rather than empirical proofs but this does not seem to have happened with there being very little, if any, improvement in this category.

This phenomenon was echoed in question G1 where the focus was on proofs of the statement "When you add the interior angles of any triangle, your answer is always 180◦ (Appendix 1, p.8, Appendix 2, p.9). This question is similar to question A1 but looked at geometric arguments and statements instead of algebraic ones. In survey 1, for 2 question parts, 13/26 responses were concerned with empirical proofs. False proofs are actually demonstrations pupils will have been shown in lessons that involve showing that the sum of the angles in a triangle equal 180˚ by tearing off the corners of a triangle and then placing them in the line. From experience of year 7 work, it seems pupils wrongly believe this is a proof of the statement and this is something very rarely remedied in lessons, further suggesting proof is not taught in schools. Pupils are developing a false mathematical concept of proof and believe verification of a few special cases is sufficient to make general claims. When asked if this particular activity shows the statement is always true 8/13 believed it did, considering this an accurate proof. As in question A1, pupils believed teachers would value algebraic or narrative answers (11/13 responses).

In question 2, pupils utilised an empirical approach but believed a teacher wanted a more rigorous analytic approach. One response (D) used an algebraic approach but is not actually correct. The expectation should be that pupils choose this question, when asked which questions would get the best mark, as it is purely algebraic. However, when confronted with another correct algebraic proof, 6 pupils picked the correct proof and 3 the incorrect one, implying pupils have improved slightly in their ability to identify correct proofs.

Question A3 from survey 2, a selection of proofs for the statement "When you multiply any three consecutive numbers, your answer is always a multiple of 6" (Appendix 2, p.6), echoes the above comments: 9/13 pupils chose an empirical approach to proving a statement but the same proportion then chose a more analytic proof for a teacher's requirements. This raises some questions regarding pupils' responses in general. Pupils often select the correct and valid proofs as the best answer or the answer they consider would get the best mark, they are not choosing this as the answer they would themselves use. Is this honesty on the part of the pupil? Are these pupils admitting their skills in proof are not the highest and are they aware they wouldn't be able to construct a formal analytic proof? This is something this assignment cannot offer a conclusion on. It does suggest that, given more work with proof, as skills increase pupils may begin to select the valid proofs as their own answers.

Question A6 from the first survey regards the statement "When you add any 3 consecutive numbers, your answer is always even" (Appendix 1, p.7) and showed pupils' initial knowledge of counter-examples. Healy and Hoyles (1998) found pupils were "very unlikely to choose a simple counter-example for best mark" (Healy and Hoyles, 1998, p.21). Results from this survey seem to contradict these findings: 6/13 pupils chose a simple counter-example as the best answer to this question. Pupils chose an algebraic statement as the answer that would give the best mark (8/13) but this still suggests pupils appreciate the power of a counter-example. This was developed in the lesson sequence, pupils seemed confident with this concept and able to use it when questioned further.

Questions A4 in survey 1 and A2 in survey 2 remained the same in order to monitor any variations. The questions asked pupils to construct a proof of their own for the statement "When you add any odd numbers, your answer is always even" (Appendix 1, p67, Appendix 2, p.5). There were a variety of answers in survey 1, 2/13 responses being a formal and valid proof. A further 6 responses were of a reasonable standard and could be developed to form a full proof. These findings correlate with those of Coe and Ruthven (1994) who talk about a 'proof opportunity', as discussed in the Literature Review. One of the purposes of the lesson sequence was to build on this idea and look at how proofs can be formed by connecting statements with logical inference. This was echoed in the work completed in the lessons; when pupils were asked to create a proof related to odd and even numbers, 12/14 students utilised a formal analytic approach. Many of these used different variables to represent different numbers, showing further appreciation of the situation and understanding of generality. Often this initial work was not developed and students seemed satisfied with a statement that was correct and then continued with another task.

Responses from survey 2 showed more pupils utilised an empirical approach to proving this statement. 8/11 responses to question A2 were of a reasonable standard and could have been developed into a full proof. This shows no real improvement in ability by pupils, suggesting further lessons may be needed to improve proving skills. In the question, the number of odd numbers to be added together is not specified, and one pupil spotted this in the second survey perhaps indicating pupils are looking in more detail at statements and being more analytical about the situation.

Question A4 of survey 2 was included to investigate how pupils would set up a proof that is possibly more difficult than previous ones; the statement to be proved is “If p and q are any two odd numbers, (p + q) x (p − q) is always a multiple of 4” (Appendix 2, p.8). Lessons covered a lot of statements on odd and even numbers and it would perhaps be beneficial to see if pupils could apply knowledge to different situations and identify important aspects of a statement, then signpost the rest of the proof. Half of the responses (4/8) utilised an analytic approach, with focus on multiplying out the brackets in the statement. No pupils saw the situation should be modelled with an odd number represented as 2n − 1. However, this does indicate a slight improvement in proof creation and suggests pupils have realised an empirical approach is not sufficient to prove a statement. Only 3 responses utilised this category of response, whereas previous to the lesson sequence this would have been much higher, pupils may have more understanding that proofs need to be generally true not just true in one situation. This further agrees with Coe and Ruthven (1994) as pupils are not fully able to develop proofs and can only form the foundations for one.

The final question of survey 1, G6, was concerned with constructing a simple proof using a diagram of an isosceles triangle and then explaining the reasons behind each step (Appendix 1, p.12). This question was answered incorrectly in 8/11 responses, showing the majority of pupils were unable to reason why the stages of a proof were correct. This is another particular skill that would be developed in the lessons; explaining each stage of thinking is important and something pupils seem unable to carry out effectively. In a similar vein, question G2 of the second survey sought to examine if pupils had improved their ability to explain the reasons behind steps of a proof by providing a diagram of an isosceles triangle and asking pupils to write a proof to show the size of a particular angle (Appendix 2, p.11). Results of this questions are not particularly conclusive as there were only 5 responses to part b. However, 4 responses were correct and gave valid reasons for the particular step of the proof. This implies pupils are now thinking about the reasons behind their mathematics and some improvement has been made. This is echoed in the work from the lessons; when asked to complete a proof frame and provide reasons for each step, 6/11 responses were either almost or fully complete, suggesting pupils are now beginning to appreciate the idea of providing reasons for every stage of a proof.

Subsequent questions in survey 2 experienced a further drop in responses and therefore provide little insight into pupils' ability of identifying or creating proof. It is not certain what caused this fall but further analysis would be futile due to such a low rate of response.

Comparison of Year 10 conceptions and ability

Healy and Hoyles (1998) found "students who regard proof as about establishing the truth of a statement are better at evaluating arguments in terms of their correctness and generality" (Healy and Hoyles, 1998, p.61). Furthermore, "students who have some idea of the role of proof are less likely to choose empirical arguments than those who do not (Healy and Hoyles, 1998, p.30). Therefore, it seems beneficial to compare pupils' responses to P1 with the rest of their survey answers.

In survey 1, 3 pupils have been considered to give a correct answer for P1 (herein known as proof-correct) and 7 stated they had no idea what proof is (herein known as proof-false). Analysing the survey in sections, a variety of claims can be made. Firstly, when taking the mean of correct values for questions A1 and A2, proof-correct pupils scored 4.3 out of 9 whereas proof-false only scored 2.3. When doing the same for questions A6, G1 and G2, proof-correct pupils scored 6.7 out of 12 and proof-false 5.1. Similar results occur when we look at question G6, with proof-correct pupils having a mean value of 2 and proof-false pupils 1.3 (out of a possible 3). Therefore for survey 1, pupils who have an accurate conception of proof seem better at identifying correct proofs and able to correctly answer questions on proof statements and the validity of proofs.

When proof constructions are analysed, the results are less obvious. In question G4, pupils were asked for a proof of the statement "If you add the interior angles of any quadrilateral, your answer is always 360°" (Appendix 1, p.11) 3 proof-false students gave an empirical answer whereas only 1 proof-correct pupil gave an empirical answer. Furthermore, a proof-correct pupil gave the only correct proof. When looking at A4, however, the results are more mixed as both a proof-false and proof-correct pupil gave 1 empirical proof. The remaining 2 proof-correct pupils did give the only correct answers for this question. Hence, for survey 1, the results seem to agree with Healy and

Hoyles (1998), pupils who have a correct conception of proof seem to be less likely to provide an empirical argument and to be more accurate in their proof constructions.

Survey 2 saw more pupils provide accurate answers to P1: there are 5 proof-correct pupils and 3 proof-false. When looking at questions A1, A3 and G1 to G4 together, we find proof-correct pupils have a mean average score of 9.4 (out of a possible 27) and proof-false a score of 5. This suggests a correspondence between proof conception and ability in identifying correct proofs. However, neither score is particularly high and no pupil scored above 50%, weakening claims about proof-correct pupils being superior. It still seems there is a positive correlation between having a more developed conception of proof and being able to identify correct proofs.

Looking at proof constructions from questions A2 and A4, we find results very similar to those from survey 1. Taking all responses together, we find both proof-correct and proof-false pupils provide 3 empirical arguments. However, proof-correct pupils provided the 2 correct proofs. Again, it seems there is a link between proof conception and construction but this link is much less developed than survey 1. Future research would perhaps look at this link in more detail; providing further lessons in proof would perhaps see this link, which depends on a more advanced knowledge of proof, emerge further.

Evaluation

This section further discusses these findings and evaluates the impacts this data has on the research questions.

Whilst this research has not reached any particularly groundbreaking or original conclusions, I believe it has affirmed research prior to the 1999 National Curriculum changes. Results tentatively point to the idea that pupils do not experience any sort of proof throughout their secondary school career. Furthermore, it seems that only a few lessons teaching proof has an effect on pupils' conceptions and ability. It seems that if proof were incorporated into the curriculum as a whole, pupils would greatly improve and progress through secondary mathematics with an appreciation of the importance of proof and its purpose in mathematics.

Main predictions of this assignment were that year 7 and 10 pupils would have similar conceptions due to a lack of teaching proof in the curriculum. This seems to have been correct as year 7 responses have more correct conceptions of proof and seem to be more open to the idea of proof and what it could possibly mean. Some responses also hint at proof being a useful and essential part of mathematics. This contrasts with year 10 pupils who had little conception of proof and were unable to connect their knowledge with anything else studied. It appears that the current mathematics curriculum has not improved pupils' knowledge of proof and could possibly have made it worse.

Another prediction was that a short sequence of lessons would improve pupil ability and conception of proof, this has been slightly successful but cannot be considered fully due to various limitations. Results of the second survey showed minor improvements in some areas, this has already been discussed in detail. However, the success was not as great as was originally hoped. Importantly, the suggestion is that teaching proof, as an important part of mathematics, will show success and pupils will realise proof is integral to the study of this area. Further research is needed to show if success in this area relates to other areas of mathematics and if teaching proof shows general improvement in all areas.

The final prediction was that pupils who have more accurate or realistic conceptions of proof perform better throughout the lessons and in both the first and second surveys, when compared with peers who had little or no knowledge or conception of proof. This proved to be partially correct, with there being a more developed link between accurate conception and identifying correct proof statements. This link became less obvious when looking at proof construction; tentative links could be suggested but further research is needed, with more lessons on proof being provided and more proof being integrated into lessons.

In conclusion, this research has laid some groundwork for further investigation. The field of knowledge could be expanded to look at what would be the best methods to teach proof to pupils, this could be incorporated when teaching proof in future research and should improve the quality of results. This would allow for further contribution to this area of research, and possible changes to the current mathematics curriculum encouraging incorporation of proof into the routine teaching of mathematics.

Implications

Even with limited data, I believe this assignment has some valuable comments to make on the teaching of proof in mathematics education. One of the flaws with the literature for this section is the majority of large scale research conducted into this area predates the described changes in the National Curriculum. The results of this assignment were hard to predict as, theoretically as it seems, pupils have followed a curriculum that places proof in a higher regard. However, this assignment suggests this is not the case as pupils seem to have had little exposure to proof. I believe pupils are not experiencing an integral part of the subject and therefore cannot fully understand the beauty and importance of mathematics.

Whilst this assignment did not set out to describe the ideal methods to utilise when teaching mathematics, I do believe it is important to mention this. Research suggests that teaching proof does have an effect but it is unclear whether providing explicit lessons with the focus on proof is ideal. This should be developed in further research as it is uncertain whether this method is best for pupils to gain an understanding of proof. It is my personal belief that the best method is to incorporate proof into lessons; skills that surround creating an accurate proof run deep throughout mathematics and the logical background of the subject is something rarely discovered by pupils. Investigational learning is now prevalent in mathematics and is a generally accepted method for effective teaching. However, investigations could be extended to venture into the realms of proof and begin to prove things absolutely, instead of showing that a pattern is intuitively true.

This assignment has also been important for my own personal development. In particular, I have learnt that an area of mathematics that is hugely emphasised at a university level is previously almost completely ignored. Skills I take for granted are not present, or so easily accessible, because pupils are not provided with opportunities to create logical chains of inference and create effective arguments. Mathematics is about communication and yet we rarely encourage pupils to express their mathematics in a way that would provide an insight into mathematics for others, not just for themselves. Allowing pupils to act as mathematicians and experience proof will encourage a different perception of mathematics and may encourage further study of a fascinating subject.

In conclusion, whilst the data does not provide an absolute statement, it does lay the groundwork for further study into this most important of topics (if it can be called that). Why is this area so important? As the first line of this assignment says "Analogy cannot serve as proof".

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Appendix 1: Proof Questionnaire 1

Proof Survey

Name:

……………………………………………………………………………………

Form:

……………………………………………………………………………………

P1:

You are going to complete a survey that is all about proof.

Before you start, write everything you know about proof in mathematics and what you think it is for.

If it helps, you can use examples, pictures or even a story: just as long as it's about proof!

A1:

Aysha, Brian, Coby, Deon, Eric and Fiona were trying to prove whether the following statement is true or false:

When you add any 2 even numbers, your answer is always even.

Aysha's answer

a is any whole number.

b is any whole number.

$2a$ and $2b$ are any two even numbers.

$2a + 2b = 2(a + b)$.

So Aysha says it's true

Brian's answer

$2 + 2 = 4$	$4 + 2 = 6$
$2 + 4 = 6$	$4 + 4 = 8$
$2 + 6 = 8$	$4 + 6 = 10$

So Brian says it's true

Coby's answer

Even numbers are numbers that can be divided by 2. When you add numbers with a common factor, 2 in this case, the answer will have the same common factor.

So Coby says it's true

Deon's answer

Even numbers end in 0, 2, 4, 6 or 8. When you add any two of these the answer will still end in 0, 2, 4, 6 or 8.

So Deon says it's true

Eric's answer

Let x = any whole number, y = any whole number.

$x + y = z$

$z - x = y$

$z - y = x$

$z + z - (x + y) = x + y = 2z$

So Eric says it's true

a) Whose answer do you like best? . . . .

b) Whose answer is closest to what you would do? . . . .

c) Whose answer would get the best mark from your teacher? . . . .

d) For each of the following, circle whether you agree, don't know, or disagree.

The statement is:

When you add any 2 even numbers, your answer is always even.

	agree	don't know	disagree
Aysha's answer ...			
shows you that the statement is **always true**	1	2	3
Brian's answer ...			
shows you that the statement is **always true**	1	2	3
Coby's answer ...			
shows you that the statement is **always true**	1	2	3
Deon's answer ...			
shows you that the statement is **always true**	1	2	3
Eric's answer ...			
shows you that the statement is **always true**	1	2	3

A2:

Suppose it has now been proved that:

When you add any 2 even numbers, your answer is always even.

Zoe asks what needs to be done to prove whether:

When you add 2 even numbers that are square, your answer is always even.

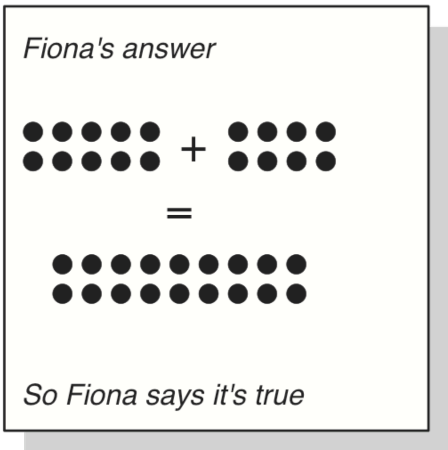
Tick (✓) either A or B.

(A) Zoe doesn't need to do anything, the first statement has already proved this. ❑

(B) Zoe needs to construct a new proof. ❑

A3:

Fiona drew the following picture for her answer to A1:



Would you choose Fiona's answer instead of your previous choice as the one closest to what you would do?

Yes ☐ **No** ☐

Would you choose Fiona's answer instead of your previous choice as the one your teacher would give the best mark?

Yes ☐ **No** ☐

For Fiona's answer, circle whether you agree, don't know, or disagree:

Fiona's answer …	agree	don't know	disagree
shows you that the statement is **always true**	1	2	3

A4:

Prove whether the following statement is true or false. Write your answer in a way that would get you as good a mark as possible.

When you add any odd numbers, your answer is always even.

A6:

Farhana, Gary, Hamble, Iris and Julie were trying to prove whether the following statement is true or false:

When you add any 3 consecutive numbers, your answer is always even.

Farhana's answer

x is any whole number.
$x + (x+1) + (x+2) = 3x + 3$
$3 + 3 = 6$
6 is divisible by 2

So Farhana says it's true.

Gary's answer

If the first number is even, then the second must be odd and the third must be even. This combination will always add up to be odd.

So Gary says it's false.

Hamble's answer

$3 + 4 + 5 = 12$
$11 + 12 + 13 = 36$
$35 + 36 + 37 = 108$
$107 + 108 + 109 = 324$

So Hamble says it's true.

Iris's answer

$2 + 3 + 4 = 9$

So Iris says it's false.

Julie's answer

Suppose first number is even, say $2x$.
$2x + (2x+1) + (2x+2) = 6x + 3$
$6x$ is even
$\therefore 6x + 3$ is odd

So Julie says it's false

a) Whose answer do you like best? ....

b) Whose answer is closest to what you would do? ....

c) Whose answer would get the best mark from your teacher? ....

G1:

Asim, Beth, Cara, Declan, Erin and Frank were trying to prove whether the following statement is true or false:

When you add the interior angles of any triangle, your answer is always 180°.

Asim's answer

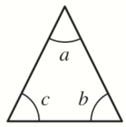
I tore the angles up and put them together.

It came to a straight line which is 180°.
I tried for an equilateral and an isosceles as well and the same thing happened.

So Asim says it's true

Beth's answer

I drew an isosceles triangle, with c equal to 65°.

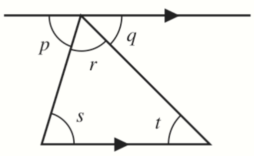


Statements	*Reasons*
$a = 180° - 2c$	Base angles in isosceles triangle equal
$a = 50°$	$180° - 130°$
$b = 65°$	$180° - (a + c)$
$c = b$	Base angles in isosceles triangle equal
$\therefore\ a + b + c = 180°$.	

So Beth says it's true

Cara's answer

I drew a line parallel to the base of the triangle.



Statements	*Reasons*
$p = s$	Alternate angles between two parallel lines are equal
$q = t$	Alternate angles between two parallel lines are equal
$p + q + r = 180°$...	Angles on a straight line
$\therefore\ s + t + r = 180°$.	

So Cara says it's true

Declan's answer

I measured the angles of all sorts of triangles accurately and made a table.

They all added up to 180°.

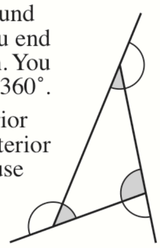
a	*b*	*c*	*total*
110	34	36	180
95	43	42	180
35	72	73	180
10	27	143	180

So Declan says it's true

Erin's answer

If you walk all the way around the edge of the triangle, you end up facing the way you began. You must have turned a total of 360°.

You can see that each exterior angle when added to the interior angle must give 180° because they make a straight line.
This makes a total of 540°.
$540° - 360° = 180°$.



So Erin says it's true

a) Whose answer do you like best? ....

b) Whose answer is closest to what you would do? ....

c) Whose answer would get the best mark from your teacher? ....

d) For each of the following, circle whether you agree, don't know, or disagree.

The statement is:

When you add the interior angles of any triangle, your answer is always 180°.

	agree	don't know	disagree
Asim's answer …			
shows you that the statement is **always true**	1	2	3
Beth's answer …			
shows you that the statement is **always true**	1	2	3
Cara's answer …			
shows you that the statement is **always true**	1	2	3
Declan's answer …			
shows you that the statement is **always true**	1	2	3
Erin's answer …			
shows you that the statement is **always true**	1	2	3

G2:

Suppose it has now been proved that:

When you add the interior angles of any triangle, your answer is always 180°.

Zak asks what needs to be done to prove whether:

When you add the interior angles of any right-angled triangle, your answer is always 180°.

Tick (✓) either A or B.

(A) Zak doesn't need to do anything, the first statement has already proved this. ☐

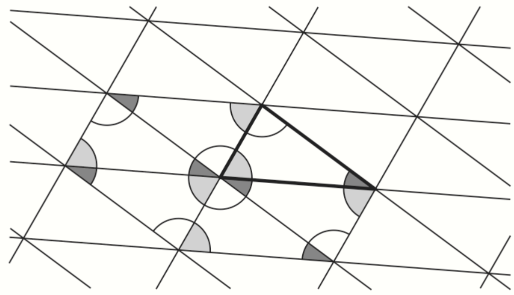
(B) Zak needs to construct a new proof. ☐

G3:

Frank gave the following answer to question G1:

Frank's answer

I drew a tessellation of triangles and marked all the equal angles.



I know that the angles round a point add up to 360°.

So Frank says it's true

Would you choose Frank's answer instead of your previous choice as the one closest to what you would do?

Yes ☐ **No** ☐

Would you choose Frank's answer instead of your previous choice as the one your teacher would give the best mark?

Yes ❑ **No** ❑

For Frank's answer, circle whether you agree, don't know, or disagree:

	agree	don't know	disagree
Frank's answer …			
shows you that the statement is **always true**	1	2	3

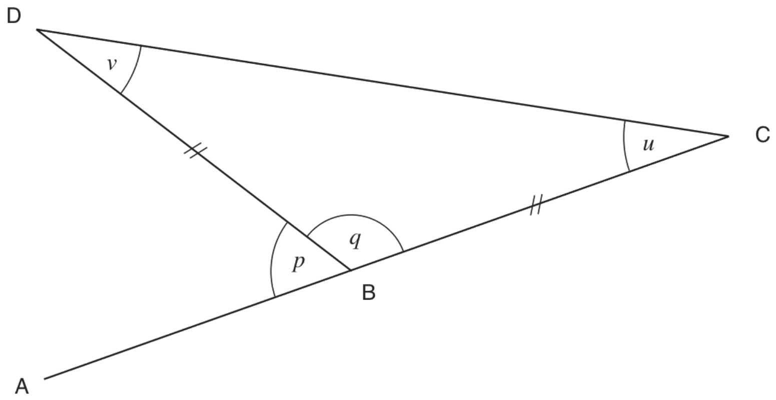
G4:

Prove whether the following statement is true or false. Write your answer in a way that would get you as good a mark as possible.

If you add the interior angles of any quadrilateral, your answer is always 360°.

G6:

In the diagram, ABC is a straight line and lines BD and BC are the same length.



a) Find the size of angle u when angle p is 70˚. ....

b) Jenny wants to find the size of angle u when angle p is 76˚.
J, K and L are her calculations, but they are not in the right order.

(J) angle u + angle v = 180˚ – 104˚ = 76˚

(K) angle u = 76˚÷ 2 = 38˚

(L) angle q = 180˚ – 76˚ = 104˚

Write the letters J, K and L in the order in which Jenny did the calculations. .. .. ..

c) These are Jenny's reasons for her calculations, but they are not in the right order either.
Match the reasons and calculations by writing the letters J, K, L in the blank circles:

( ) The base angles of an isosceles triangle are equal

( ) Angles on a straight line add up to 180˚

( ) The angle sum of a triangle is 180˚

Z1:

a. What did you feel about taking part in this survey?

b. Which question did you like best, and why?

c. Which question did you like least, and why?

d. Please add any other comments, if you wish to, about the survey?

Appendix 2: Proof Questionnaire 2

Proof Survey 2

Name:

……………………………………………………………………………………

Form:

……………………………………………………………………………………

P1:

Over the last few lessons you have been learning about proof and it's place in mathematics. This survey is your chance to write about what you now know about proof.

Like in the last survey, I would like you to write everything you know about proof in mathematics and what you think its purpose is.

A1:

The drawing shows the calender for last July.

A square is drawn around nine of the numbers.

The top-left number and bottom-right number in the square are circled.

Ashok, Beryl, Cora, Dave and Ethan are discussing whether this statement is true:

When there are nine numbers in the square, the bottom-right number will be 16 more than the top-left number.

July

M	T	W	T	F	S	S
					1	2
3	4	5	6	7	8	9
10	(11)	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	(27)	28	29	30
31						

Ashok's answer

5			
	13		
		21	
			29

Look at a 'diagonal' line like this one. You can see that the numbers increase by 8 each time.

So if you go from one diagonal number to the next-but-one diagonal number, it increases by 8 + 8 which is 16.

So Ashok says it's true

Beryl's answer

For a square with 9 numbers, you can get from the first circled number to the second by going 2 steps across and 2 steps down.

Each step across is an increase of 1 day. Each step down is an increase of 1 week.

So altogether, the number increases by 1 + 1 + 7 + 7, which is 16.

So Beryl says it's true

Cora's answer

Think of a square of nine numbers. Call the first number n.

Then this shows the first row and column. The numbers go up by a day at a time in each row and by a week at a time in each column.

n	n+1	n+1+1
n+7	-	-
n+7+7	-	-

So the last number is $n + 1 + 1 + 7 + 7$, which is $n + 16$.

So Cora says it's true

Dave's answer

It works for the original square because 27 is 16 more than 11.

(11)	12	13
18	19	20
25	26	(27)

It also works for this square, because 21 is 16 more than 5.

(5)	6	7
12	13	14
19	20	(21)

So Dave says it's true

Ethan's answer

Draw a square full of numbers.
Let x be the first number in the square.
Let d be the number of days in a week.
Let s be the number of numbers in the square.

Then the last number in the square is $x + d + s$.

But $d = 7$ and $s = 9$, so the last number is $x + 7 + 9$ which is $x + 16$.

So Ethan says it's true

a) Whose answer do you like best? . . . .

b) Whose answer is closest to what you would do? . . . .

c) Whose answer would get the best mark from your teacher? . . . .

d) For each of the following, circle whether you agree, don't know, or disagree.

The statement is: **When there are nine numbers in the square, the bottom-right number will be 16 more than the top-left number**.

Ashok's answer ...	agree	don't know	disagree
shows you that the statement is **always true**	1	2	3
only shows you that the statement is true for **some** examples	1	2	3
shows you **why** the statement is true	1	2	3

Beryl's answer ...			
shows you that the statement is **always true**	1	2	3
only shows you that the statement is true for **some** examples	1	2	3
shows you **why** the statement is true	1	2	3

Cora's answer ...			
shows you that the statement is **always true**	1	2	3
only shows you that the statement is true for **some** examples	1	2	3
shows you **why** the statement is true	1	2	3

Dave's answer ...			
shows you that the statement is **always true**	1	2	3
only shows you that the statement is true for **some** examples	1	2	3
shows you **why** the statement is true	1	2	3

Ethan's answer ...			
shows you that the statement is **always true**	1	2	3
only shows you that the statement is true for **some** examples	1	2	3
shows you **why** the statement is true	1	2	3

A2:

Prove whether the following statement is true or false. Write your answer in a way that would get you as good a mark as possible.

When you add any odd numbers, your answer is always even.

A3:

Kate, Leon, Maria and Nisha were asked to prove whether the following statement is true or false:

When you multiply any three consecutive numbers, your answer is always a multiple of 6.

Kate's answer

A multiple of 6 must have factors of 3 and 2.
If you have three consecutive numbers, one will be a multiple of 3 as every third number is in the three times table.
Also, at least one number will be even and all even numbers are multiples of 2.
If you multiply the three consecutive numbers together the answer must have at least one factor of 3 and one factor of 2.

So Kate says it's true.

Leon's answer

$1 \times 2 \times 3 = 6$

$2 \times 3 \times 4 = 24$

$4 \times 5 \times 6 = 120$

$6 \times 7 \times 8 = 336$

So Leon says it's true.

Maria's answer

x is any whole number

$$x \times (x+1) \times (x+2) = (x^2 + 2) \times (x+2)$$
$$= x^3 + x^2 + 2x^2 + 2x$$

Cancelling the x's gives $1 + 1 + 2 + 2 = 6$

So Maria say it's true.

Nisha's answer

Of the three consecutive numbers, the first number is either:
EVEN which can be written $2a$ (a is any whole number) or,
ODD which can be written $2b - 1$ (b is any whole number).

If EVEN
$2a \times (2a+1) \times (2a+2)$ is a multiple of 2.
and either a is a multiple of 3 DONE
or a is not a multiple of 3
$\therefore$ $2a$ is not a multiple of 3
$\therefore$ Either $(2a+1)$ is a multiple of 3 or $(2a+2)$ is a multiple of 3 DONE

If ODD
$(2b-1) \times 2b \times (2b+1)$ is a multiple of 2
and either b is a multiple of 3 DONE
or b is not a multiple of 3
$\therefore$ 2b is not a multiple of 3
$\therefore$ Either $(2b-1)$ is a multiple of 3 or $(2b+1)$ is a multiple of 3 DONE

So Nisha says it's true.

a) Whose answer do you like best? . . . .

b) Whose answer is closest to what you would do? . . . .

c) Whose answer would get the best mark from your teacher? . . . .

d) For each of the following, circle whether you agree, don't know or disagree.

The statement is:

When you multiply any three consecutive numbers, your answer is always a multiple of 6.

	agree	don't know	disagree
Kate's answer:			
Has a **mistake** in it	1	2	3
Shows that the statement is **always true**	1	2	3
Only shows the statement is true for **some** consecutive numbers	1	2	3
Shows you **why** the statement is true	1	2	3
Is an easy way to **explain** to someone in your class who is unsure	1	2	3
Leon's answer:			
Has a **mistake** in it	1	2	3
Shows that the statement is **always true**	1	2	3
Only shows the statement is true for **some** consecutive numbers	1	2	3
Shows you **why** the statement is true	1	2	3
Is an easy way to **explain** to someone in your class who is unsure	1	2	3
Maria's answer:			
Has a **mistake** in it	1	2	3
Shows that the statement is **always true**	1	2	3
Only shows the statement is true for **some** consecutive numbers	1	2	3
Shows you **why** the statement is true	1	2	3
Is an easy way to **explain** to someone in your class who is unsure	1	2	3
Nisha's answer:			
Has a **mistake** in it	1	2	3
Shows that the statement is **always true**	1	2	3
Only shows the statement is true for **some** consecutive numbers	1	2	3
Shows you **why** the statement is true	1	2	3
Is an easy way to **explain** to someone in your class who is unsure	1	2	3

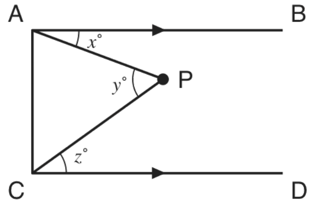
A4:

Prove whether the following statement is true or false. Write your answer in a way that would get you as good a mark as possible.

If *p* and *q* are any two odd numbers, *(p + q) x (p – q)* is always a multiple of 4.

G1:

In the diagram, line AB is parallel to line CD, and AC is at right angles to both lines.

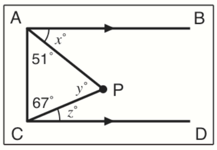


Points A, B, C and D are fixed.
Point P can move anywhere between AB and CD, but stays connected to A and C (the straight lines PA and PC can stretch or shrink).

Astrid, Burt, Cleo, Dilip and Emma are discussing whether this statement is true:

$x° + z°$ is equal to $y°$.

Astrid's answer



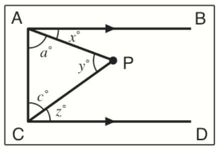
I could have a triangle APC with these angles. →

Then
$y = 180 - 51 - 67 = 62$,
$x = 90 - 51 = 39$, and $z = 90 - 67 = 23$.

But $62 = 39 + 23$, and as
$180 - 51 - 67 = (90 - 51) + (90 - 67)$,
I could have a triangle with other angles.
So $y = x + z$.

So Astrid says it's true

Burt's answer



The angle sum of triangle ACP is 180°, so $y + a + c = 180$.

Angles A and C are 90°, so I can write $90 - x$ for a, and $90 - z$ for c.

So $y + (90 - x) + (90 - z) = 180$,
so $y - x - z = 0$,
so $y = x + z$.

So Burt says it's true

Cleo's answer

I measured the angles in the original diagram. I then moved P to another place and measured the angles again.

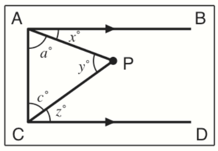
I made this table:

x	z	y
21	36	57
17	32	49

So both times I found that $x + z$ equals y.

So Cleo says it's true

Dilip's answer



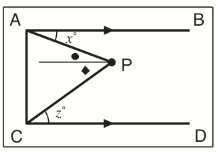
The angle sum of triangle ACP is 180°.

So I can write $a + c = 180 - y = 180 - (x + z)$.
Also $y = 180 - (a + c)$.

So $y = 180 - (180 - (x + z)) = x + z$.

So Dilip says it's true

Emma's answer



I drew a line through P parallel to lines AB and CD.

The new line cuts angle y into two parts.
The top part (●) is equal to x because the new line is parallel to AB. The bottom part (◆) is equal to z because the new line is parallel to CD.
So, altogether, y is equal to $x + z$.

So Emma says it's true

a) Whose answer do you like best? ....

b) Whose answer is closest to what you would do? ....

c) Whose answer would get the best mark from your teacher? ....

d) For each of the following, circle whether you agree, don't know, or disagree.

The statement is: $x^\circ + z^\circ$ **is equal to** y°.

Astrid's answer ...	agree	don't know	disagree
shows you that the statement is **always true**	1	2	3
only shows you that the statement is true for **some** examples	1	2	3
shows you **why** the statement is true	1	2	3

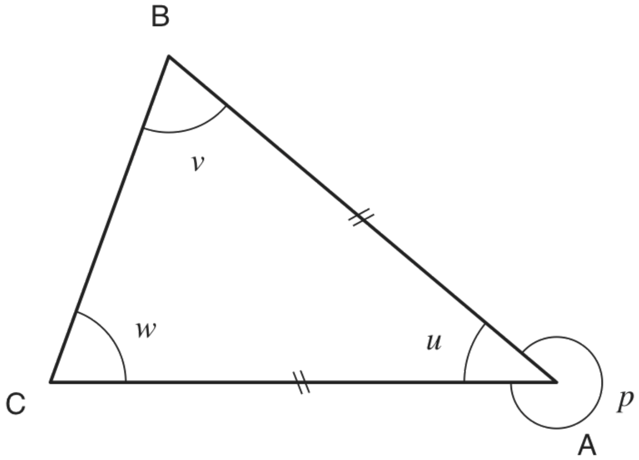
Burt's answer ...			
shows you that the statement is **always true**	1	2	3
only shows you that the statement is true for **some** examples	1	2	3
shows you **why** the statement is true	1	2	3

Cleo's answer ...			
shows you that the statement is **always true**	1	2	3
only shows you that the statement is true for **some** examples	1	2	3
shows you **why** the statement is true	1	2	3

Dilip's answer ...			
shows you that the statement is **always true**	1	2	3
only shows you that the statement is true for **some** examples	1	2	3
shows you **why** the statement is true	1	2	3

Emma's answer ...			
shows you that the statement is **always true**	1	2	3
only shows you that the statement is true for **some** examples	1	2	3
shows you **why** the statement is true	1	2	3

G2:



The diagram shows a triangle ABC.
Side AB is the same length as side AC.

a) Find the size of angle v, when angle p is 320˚. . . . .

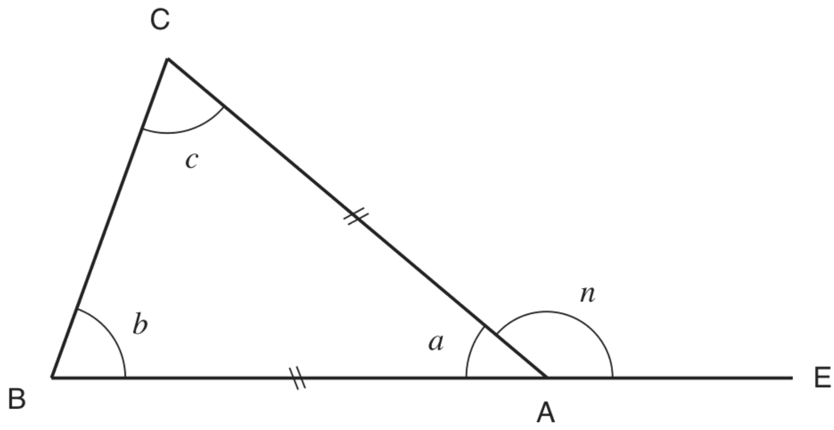
Write down each step of your calculation.

b) Write down your first step again and give a reason for the step.

c) Write down your next steps again and give a reason for each one.

G3:

This diagram shows
a triangle ABC.
Side AB is the same
length as side AC.
Line BAE is straight.



a) Find the value of c when $n = 140^\circ$. . . . .

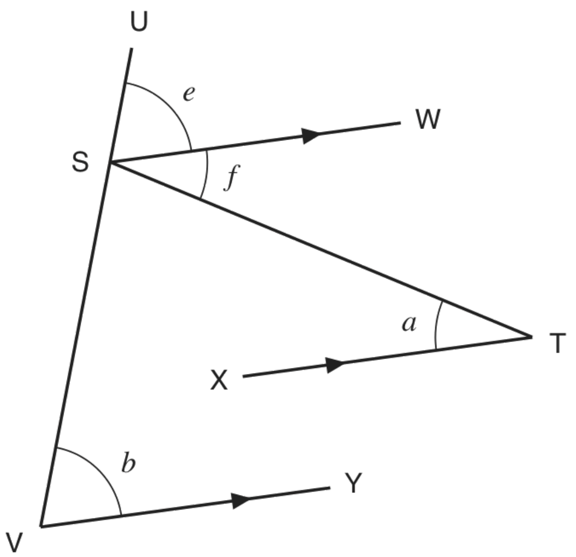
Write down each step of your calculation.

b) Show that $c = \frac{1}{2}n$, whatever the value of n.

Write down all your steps.

G4:

c) In this diagram, lines SW, XT and VY are parallel.
Line USV is straight.



Show that $a = U\hat{S}T - b$.

Write down all your steps.

G5:

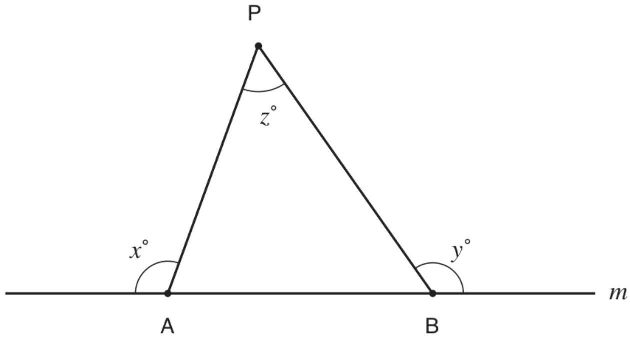
Prove whether the following statement is true or false. Write your answer in a way that would get you as good a mark as possible.

If you add the interior angles of any quadrilateral, your answer is always 360°.

G6:

In the diagram, A and B are two fixed points on a straight line m.

Point P can move, but stays connected to A and B (the straight lines PA and PB can stretch or shrink).



Avril, Bruno, Chandra and Don are discussing whether this statement is true:

$x^\circ + y^\circ$ is equal to $180^\circ + z^\circ$.

Avril's answer

I measured the angles in the diagram and found that angle x is 110°, angle y is 125° and angle z is 55°.

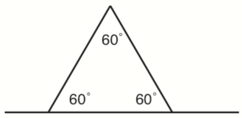
$110^\circ + 125^\circ = 235^\circ$,

and $180^\circ + 55^\circ = 235^\circ$.

So Avril says it's true

Bruno's answer

I can move P so that the triangle is equilateral, and its angles are 60°.



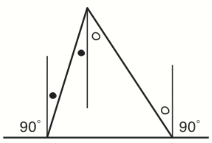
So x is 120° and y is 120°.

$120^\circ + 120^\circ$ is the same as $180^\circ + 60^\circ$.

So Bruno says it's true

Chandra's answer

I drew three parallel lines.
The two angles marked with a ● are the same and the two marked with a ○ are the same.



Angle x is 90°+ ● and angle y is 90°+ ○.

So x plus y is 180 + ● + ○, which is 180 + z.

So Chandra says it's true

Don's answer

I thought of a diagram where the angles x, y and z are all 170°.

So in my diagram $x + y$ is not equal to $180 + z$.

So Don says it's not true

a) Whose answer is closest to what you would do? . . . .

b) Whose answer would get the best mark from your teacher? . . . .

For each of the following, circle whether you agree, don't know, or disagree.

The statement is:
$x^\circ + y^\circ$ is equal to $180^\circ + z^\circ$.

Avril's answer ...	agree	don't know	disagree
shows you that the statement is **always true**	1	2	3
only shows you that the statement is true for **some** examples	1	2	3
shows you **why** the statement is true	1	2	3

Bruno's answer ...	agree	don't know	disagree
shows you that the statement is **always true**	1	2	3
only shows you that the statement is true for **some** examples	1	2	3
shows you **why** the statement is true	1	2	3

Chandra's answer ...	agree	don't know	disagree
shows you that the statement is **always true**	1	2	3
only shows you that the statement is true for **some** examples	1	2	3
shows you **why** the statement is true	1	2	3

Don's answer ...	agree	don't know	disagree
shows you that the statement is **not true**	1	2	3
shows you **why** the statement is not true	1	2	3

Z1:

a. What did you feel about taking part in this survey?

b. Which question did you like best, and why?

c. Which question did you like least, and why?

d. Please add any other comments, if you wish to, about the survey?

Appendix 3: Questionnaire Statement

Statement to be read out before conducting the Questionnaires (based on Healy and Hoyles, 1998):

This questionnaire is part of some research I am conducting into students' views on proof and what their abilities are. However, this is not a test and everyones identity will remain confidential.

I am interested in your individual views so you will need to complete this on your own and if there is anything you don't understand or can't do, write everything you can think of and move on.

Most of the questions are structured so that there is a mathematical statement followed by a number of answers given by students who were trying to work out whether the statement was true or false. Some of these proofs are correct and some are false but there is never only one right answer.

Other questions are a bit different and involve you just circling a letter or writing your own proofs. There are always instructions at the start of the question so read these carefully. If you are stuck, let me know and I can clarify what the question is asking.

When you are writing your own proofs, you can base your answers on examples in the survey or you can come up with new answers of your own.

If you need any equipment, let me know but you should just need a pencil or pen. Please do any writing or notes on the survey, if you need more space, just use the back of the sheet.

Appendix 4: Research Code of Practice

Code of practice (based on Bell, p.51):

Informal discussion with mentor to obtain agreement for the study

Refinement of the research questions and the methods to be used

Discussion with mentor and class teachers about methods to be used

Adjustments made to the methods in accordance with advice

All participants will be offered the opportunity to remain anonymous

All information will be treated with the strictest confidentiality

All participants will be able to access the final assignment

The research will attempt to explore the teaching of proof to secondary school students. I hope that the final assignment will be of benefit to the school and those teachers whose classes I will work with.

Appendix 5: Year 10 Survey 1 Data

This section contains 2 exemplar pages (pages 1 and 2 of 6) from the data recording methods used.

In this section, the pupil responses are colour coded. Grey answers mean no answer was given, green answers can be considered as correct answers, amber answers mean that the response was close to being complete but more work is needed, red answers are answers that are either incorrect or responses can be considered to be incorrect or invalid.

	P1	A1 a	A1 b
Pupil 1	"Isn't it just demonstrating that a formula or rule works?"	C	B
Pupil 2	"I don't know"	D	D
Pupil 3	"I don't know what it is"	D	A
Pupil 4	"I think that proof is when people ask you to proove something (but that is just a guess)!"	C	B
Pupil 5	"I don't know anything about proof, I'm sorry."	B	B
Pupil 6	"I have no clue what proof and all that is."	B	A
Pupil 7	"I have never heard of 'proof' in maths. Unless all it means is proving that something is true or false?"	C	D
Pupil 8	"Is it where you have to prove stuff is true?"	B	C
Pupil 9	"I don't know what it is."	B	E
Pupil 10	"I am unsure what proof is!" "Proof is showing that a statement is always true"	A	A
Pupil 11	"I'm not sure what it is"	E	A
Pupil 12			
Pupil 13	"I don't know what it is" "After doing it, I have some idea of what it is."	D	B
Pupil 14	"Proof is something that can be scientifically or mathematically shown consistantly as evidence behind something else."	C	A

A1 c	A1 d i	A1 d ii	A1 d iii	A1 d iv	A1 d v	A2	A3 a	A3 b
C	2	3	1	1	2	A	Y	N
E	2	2	2	2	2	A	N	N
E	2	2	1	1	2	B	N	N
E	1	3	1	1	2	A	N	N
B	1	1	1	1	1	A	N	Y
A	1	1	1	1	1	A	N	N
A	1	3	1	1	2	A	N	N
E	1	2	3	2	1	A	Y	N
E	2	2	2	2	2	A	N	Y
E	1	3	2	2	1	B	N	N
E	1	3	2	2	1	B	N	N
D	2	2	2	1	2	A	N	N
C	1	1	1	2	1	A	N	N

Appendix 6: Year 10 Survey 2 Data

This section contains 2 exemplar pages (pages 1 and 2 of 6) from the data recording methods used.

In this section, the pupil responses are colour coded. Grey answers mean no answer was given, green answers can be considered as correct answers, amber answers mean that the response was close to being complete but more work is needed, red answers are answers that are either incorrect or responses can be considered to be incorrect or invalid.

A2	A3 a	A3 b	A3 c	A3 d i
Analytic formal explanation that reaches a conclusion of 2(n+m+1), so as there is a factor of 2, this must be even.	K	L	N	2 2 2 2 2
Claim that the statement is false and an analytic formal explanation of 2n+2n+1 but no further explanation	L	L	N	2 2 2 2 2
Empirical explanation showing that 3+5 = 8.	N	N	N	1 1 1 1 1
Analytic formal explanation that reaches 4m+2. No further explanation given.	K	L	M	2 1 3 1 1
Empirical explanation by adding 5 pairs of the same odd numbers and then an exhaustive idea that all odd numbers end in these digits.	L	L	M	3 2 1 2 3
Visual diagram using a dot representation for 7 + 5 = 12	K	L	N	1 3 1 3 1
Assertion that the statement is false; it is true if you add an even number of odd number but false otherwise. Then a visual diagram to show this.	M	M	N	1 3 2 3 1
Empirical explanation showing that 5+5 = 10.	K	L	N	2 1 2 3 2
Empirical expanation by looking at two pairs of the same number. Then an analytic formal explanation using (2m+1)+(2m+1) with no further development.	K	L	M	2 1 2 1 3
---	K	L	N	2 1 3 3 1
Analytic formal explanation that reaches 4m+2. No further explanation given.	L	L	N	2 1 3 1 3
Empirical explanation using 1+3 = 4. Then a narrative explanation using dots and explaining how 2 odd numbers pair up single dots to form an even number.	K	K	N	2 1 3 1 1
---	K	K	K	3 1 3 1 1

A3 d ii	A3 d iii	A3 d iv	A4	G1 a
2 3 1 3 2	2 3 2 3 2	2 1 1 1 3	Analytic explanation that we have $p^2 - q^2$. No further explanation given.	C
2 2 2 2 2	2 2 2 2 2	2 2 2 2 2	Claim that the statement is false and then the empirical explanation of 3x3=9.	C
1 1 1 1 1	3 3 3 3 3	1 1 1 1 1	---	E
3 3 1 3 1	3 1 3 1 3	1 1 3 1 3	---	A
2 3 1 2 1	1 3 3 2 3	2 2 3 2 3	Empirical explanation using 2 pairs of numbers and showing that their answers are multiples of 4.	D
3 3 1 1 1	2 1 3 1 3	3 1 3 1 3	Empirical explanation using 2 pairs of numbers and showing that their answers are multiples of 4.	---
3 2 2 3 1	1 1 3 3 2	1 1 2 2 3	Assertion that $p^2 - q^2$ but no further explanation given.	E
2 3 2 1 2	2 1 2 3 2	2 3 2 1 2	---	E
3 3 1 3 3	2 1 3 1 2	2 1 1 1 3	Empirical explanation using 1 pair of 2 numbers and showing that the answer is a multiple of 4.	---
1 3 3 1 1	2 1 3 1 3	2 1 3 1 3	---	C
2 3 1 3 1	2 1 3 3 1	2 1 3 1 3	Attempt at analytic formal explanation but multiplication not completed.	D
3 3 1 3 3	2 2 2 2 3	2 2 2 2 3	---	C
1 3 1 3 2	1 3 3 1 2	2 2 2 2 2	Analytic explanation that we have $p^2 - q^2$. Incorrect empirical explanation using 3 as both p and q and stating that 18 is not a multiple of 4.	---